Analytic eigenenergies of the Dirac equation with finite degrees of freedom under a confining linear potential using basis functions localized in spacetime

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Considering the propagation of fields in the spacetime continuum and the well-defined features of fields with finite degrees of freedom, the wave function is expanded in terms of a finite set of basis functions localized in spacetime. This paper presents the analytic eigenenergies derived for a confined fundamental fermion-antifermion pair under a linear potential obtained from the Wilson loop for the non-Abelian Yang-Mills field. The Hamiltonian matrix of the Dirac equation is analytically diagonalized using basis functions localized in spacetime. The squared lowest eigenenergy (as a function of the relativistic quantum number when the rotational energy is large compared to the composite particle masses) is proportional to the string tension and the absolute value of the Dirac's relativistic quantum number related to the total angular momentum, consistent with the expectation.

1 Introduction

In the formalism elaborated in our previous publications [1, 2], the fields are expanded in terms of basis functions localized in spacetime. A key characteristic of this formulation based on finite element theory is that it is possible to apply differentiation unlike in the finite difference method. Our method allows the use of a basis set of step functions, which is rather different from the formulation by Bender *et al.* [3]. In the non-Abelian Yang-Mills case [4-7], the analytic continuum classical field as a possible vacuum reveals the linear potential, and quantum fluctuations are expressed in terms of step functions exhibiting Coulomb potential. Regarding the confined bound state of a fundamental fermion-antifermion pair and related themes, one can observe approaches from field theory and lattice gauge theory in prior literature [8-18]. Although the Dirac equation with a linear potential was investigated analytically by other authors [19-22], the Regge trajectory [23] may not be reproduced systematically in the relativistic scheme. Concerning a classical mechanical Hamiltonian that describes the principal properties of the Regge trajectory, comprising a linear potential and repulsive rotational potential [24, 25], which is to be described in the part of this paper containing Eqs. (57)-(62) of Subsection 3.1, the Hamiltonian has no basis in the theory of Dirac fields with the potential produced by the Yang-Mills fields. Furthermore, a clear answer has not been fully provided at the quantum level by other theoretical/numerical approaches to the following questions. What is the mechanism of the mass of a pair composed of a fundamental fermion and antifermion? Why can the binding energy (mass) of the paired fermions be large compared with the masses of the composite fermion and antifermion? The confined bound state of composite fermions is not fully understood theoretically. Because the Regge trajectory is expressed by the quantum number corresponding to the rotational quantity, it is expected that the Dirac fields, expressed in spherical coordinates with the potential derived directly from the Yang-Mills theory, will provide answer to the aforementioned questions and reveal the meaning of the classical mechanical Hamiltonian.

This work is aimed at obtaining an analytic expression for the eigenenergies of a confined fundamental fermion-antifermion pair using our formalism mentioned above [1, 2], which formulates fields of finite degrees of freedom using basis functions localized in the spacetime continuum. The present formalism enables an analytical calculation without using numerical values unlike numerical computer simulations via numerical values. The action-like total Hamiltonian including a given linear potential, which leads to the Dirac equation in spherical coordinates by variational calculus, is expressed in terms

of a basis set of step functions localized in spacetime. The Hamiltonian matrix in the secular equation is diagonalized analytically, and the lowest eigenvalue is derived as a function of the string tension and Dirac's relativistic quantum number related to the total angular momentum. The squared system energies for the large rotational energy compared to the constituent particle masses correspond to those in the classical mechanical Hamiltonian case, which are consistent with the principal properties of the Regge trajectory [23]. We emphasize that (1) the analytical approach shows clearly that the squared system energies are proportional to the string tension and quantum number, and that they originate from the secular equation structure leading to the absolute value of the quantum number that includes a sign and constructs the potential; (2) the quantum number, instead of being a non-relativistic value, is the relativistic integer described by the Dirac equation for a pair of fundamental fermions.

This article is organized as follows. Section 2 describes the formalism for the Dirac equation using a basis set of step functions localized in spacetime. Section 3 presents an analytic expression for the eigenenergies of a confined bound fermion-antifermion pair, including discussions in Subsection 3.2, and is followed by conclusions in Section 4.

2 Basis equations and theoretical formalism

2.1 Dirac equation for the non-Abelian case in spherical coordinates

To avoid confusion, we first note that the relativistic κ used in this paper has the following relation [26] with the Dirac's notation $j_{\rm D}$ [27]

$$\kappa = j_{\rm D} = \begin{cases} l+1 & \text{for } j = l+1/2 \\ \\ -l & \text{for } j = l-1/2 \end{cases}, \tag{1}$$

where l and j refer to the quantum numbers for angular and total angular momentums, respectively. For the lowest energy case of a hydrogen atom, $\kappa > 0$. Denoting the radial wave functions of fermions in spherical coordinates with radial r-axis as

$$\psi_F(r) = \frac{F(r)}{r}, \tag{2}$$

$$\psi_G(r) = \frac{G(r)}{r}, \tag{3}$$

the Dirac equations in natural units become

$$(+m - \frac{\alpha}{r})F - \frac{dG}{dr} - \frac{\kappa}{r}G = EF, \tag{4}$$

$$(-m - \frac{\alpha}{r})G + \frac{dF}{dr} - \frac{\kappa}{r}F = EG, \tag{5}$$

where m is the fermion mass and E is the fermion energy.

Considering the forms of $\psi_F = F/r$ and $\psi_G = G/r$ as well as the two-dimensional integral $\int dr\, r^2$, the total Hamiltonian that variationally leads to the above Dirac equations is

$$\begin{aligned} \mathcal{H} = \frac{1}{2}\int dr[&F(+m - \frac{\alpha}{r})F - F\frac{dG}{dr} - F\frac{\kappa}{r}G \\ &+ G(-m - \frac{\alpha}{r})G + G\frac{dF}{dr} - G\frac{\kappa}{r}F \\ &- FEF - GEG]. \end{aligned} \tag{6}$$

We now add the linear energy potential derived from the Wilson loop for a non-Abelian field. Energy is one component of the four-vector momentum and the linear potential constructs the energy potential combined consistently with the Coulomb potential, the Coulomb potential $-\alpha/r$ above is replaced by $-\alpha/r + \sigma r$. Here, $\alpha = g^2/(4\pi)$ with a coupling constant g, and σ refers to the string tension. We then have the total Hamiltonian for the non-Abelian case

$$\begin{aligned} \mathcal{H} = \frac{1}{2}\int dr[&F(+m - \frac{\alpha}{r} + \sigma r)F - F\frac{dG}{dr} - F\frac{\kappa}{r}G \\ &+ G(-m - \frac{\alpha}{r} + \sigma r)G + G\frac{dF}{dr} - G\frac{\kappa}{r}F \\ &- FEF - GEG]. \end{aligned} \tag{7}$$

2.2 Fields expanded in terms of basis functions localized in spacetime

The present formalism for fields is firstly based on the propagation of fields in the spacetime continuum, and secondly, on the fact that fields are definite in a scheme of finite degrees of freedom. We then expand the fields in terms of basis functions localized in the spacetime continuum, which has a finite number of lattice (grid) points, by realizing the following formulation. For the considering region in spherical coordinates, we introduce lattice (grid) points r_n $(n = 1, 2, ..., N)$ in the radial r-axis. The infinitesimal positive Δ is defined by $\Delta = r_{n+1} - r_n$, and $r_{n-1/2} = r_n - (1/2)\Delta$ and $r_{n+1/2} = r_n + (1/2)\Delta$. The basis functions, which have a superscript without and with prime, are defined by

$$\Omega_n^E(r) = \begin{cases} 1 & \text{for } r_{n-1/2} < r < r_{n+1/2} \\ \\ 0 & \text{for } r \le r_{n-1/2} \text{ or } r \ge r_{n+1/2} \end{cases}, \tag{8}$$

$$\Omega_n^{E'}(r) = \begin{cases} 1 & \text{for } r_{n-1/2} \le r \le r_{n+1/2} \\ \\ 0 & \text{for } r < r_{n-1/2} \text{ or } r > r_{n+1/2} \end{cases}, \tag{9}$$

which have the derivatives

$$\frac{d\Omega_n^E(r)}{dr}|_{r=r_{n-1/2}} = \frac{d\Omega_n^{E'}(r)}{dr}|_{r=r_{n-1/2}}$$

$$= \delta(r - r_{n-1/2}), \tag{10}$$

$$\frac{d\Omega_n^E(r)}{dr}|_{r=r_{n+1/2}} = \frac{d\Omega_n^{E'}(r)}{dr}|_{r=r_{n+1/2}}$$

$$= -\delta(r - r_{n+1/2}). \tag{11}$$

The above basis functions have the following properties

$$\int dr\Omega_n^E(r)\Omega_k^E(r) = \delta_{nk}\int dr\Omega_n^E(r) = \Delta\delta_{nk}, \tag{12}$$

$$\int dr\Omega_n^E(r)\Omega_k^{E'}(r) = \delta_{nk}\int dr\Omega_n^E(r) = \Delta\delta_{nk}, \tag{13}$$

$$\int dr\Omega_n^{E'}(r)\Omega_k^{E'}(r) = \delta_{nk}\int dr\Omega_n^{E'}(r) = \Delta\delta_{nk}, \tag{14}$$

where δ_{nk} is Kronecker's delta, and considering the overlap between the basis function Ω_n^E (or $\Omega_n^{E'}$) around the point r_n and the delta function around $r_{k-1/2}$ and $r_{k+1/2}$, which is the derivative of the basis function around r_k,

$$\int dr\Omega_n^E(r)\frac{d\Omega_k^E(r)}{dr} = \int dr\Omega_n^E(r)\frac{d\Omega_k^{E'}(r)}{dr}$$

$$= \int dr\Omega_n^E(r)[\delta(r - r_{k-1/2}) - \delta(r - r_{k+1/2})] = 0, \tag{15}$$

$$\int dr\Omega_n^{E'}(r)\frac{d\Omega_k^E(r)}{dr} = \int dr\Omega_n^{E'}(r)\frac{d\Omega_k^{E'}(r)}{dr}$$

$$= \int dr\Omega_n^{E'}(r)[\delta(r - r_{k-1/2}) - \delta(r - r_{k+1/2})]$$

$$= -\delta_{k,n-1} + (\delta_{k,n} - \delta_{k,n}) + \delta_{k,n+1}$$

$$= \delta_{k,n+1} - \delta_{k,n-1}. \tag{16}$$

Using Eqs. (8) and (9), we further define

$$\Omega_n^e(r) = \frac{1}{2}[\Omega_n^E(r) + \Omega_n^{E'}(r)]. \tag{17}$$

With the help of Eqs. (8)-(17), we get

$$\int dr\Omega_n^e(r)\Omega_k^e(r)$$

$$= \frac{1}{4}\int dr(\Omega_n^E\Omega_k^E + \Omega_n^{E'}\Omega_k^E + \Omega_n^E\Omega_k^{E'} + \Omega_n^{E'}\Omega_k^{E'})$$

$$= \frac{4}{4}\delta_{nk}\Delta = \Delta\delta_{nk}, \tag{18}$$

$$\int dr\Omega_n^e(r)\frac{d\Omega_k^e(r)}{dr}$$

$$= \frac{1}{4}\int dr(\Omega_n^E\frac{d\Omega_k^E}{dr} + \Omega_n^E\frac{d\Omega_k^{E'}}{dr} + \Omega_n^{E'}\frac{d\Omega_k^E}{dr} + \Omega_n^{E'}\frac{d\Omega_k^{E'}}{dr})$$

$$= \frac{1}{4}[0 + 0 + (\delta_{k,n+1} - \delta_{k,n-1}) + (\delta_{k,n+1} - \delta_{k,n-1})]$$

$$= \Delta\frac{(\delta_{k,n+1} - \delta_{k,n-1})}{2\Delta}. \tag{19}$$

Then, the expansion of the fermion wave functions

$$F(r) = \sum_n F_n\Omega_n^e(r), \tag{20}$$

$$G(r) = \sum_n G_n\Omega_n^e(r), \tag{21}$$

uniquely discretizes the following terms in the total Hamiltonian

$$\int drF(r)G(r)$$

$$= \int dr\sum_n\sum_k[F_n\Omega_n^e(r)G_k\Omega_k^e(r)]$$

$$= \sum_n\sum_k[F_nG_k\int dr\Omega_n^e(r)\Omega_k^e(r)]$$

$$= \sum_n\sum_k[F_nG_k\Delta\delta_{nk}] = \Delta\sum_n F_nG_n, \tag{22}$$

for which we have used Eq. (18).

Similarly, using Eq. (19) it follows that

$$\int drF(r)\frac{dG(r)}{dr}$$

$$= \int dr\sum_n\sum_k[F_n\Omega_n^e(r)G_k\frac{d\Omega_k^e(r)}{dr}]$$

$$= \sum_n\sum_k[F_nG_k\Delta\frac{(\delta_{k,n+1} - \delta_{k,n-1})}{2\Delta}]$$

$$= \Delta\sum_n F_n\frac{G_{n+1} - G_{n-1}}{2\Delta}. \tag{23}$$

2.3 Matrix form of Dirac equation in terms of basis functions localized in spacetime

In our scheme, the aforementioned total Hamiltonian in a fermion-confined region is expressed by

$$\mathcal{H} = \frac{\Delta}{2}$$

$$\times \sum_n \{[F_n(+m - \frac{\alpha}{r_n} + \sigma r_n)F_n$$

$$- F_n \frac{G_{n+1} - G_{n-1}}{2\Delta} - F_n \frac{\kappa}{r_n} G_n]$$

$$+ [G_n(-m - \frac{\alpha}{r_n} + \sigma r_n)G_n$$

$$+ G_n \frac{F_{n+1} - F_{n-1}}{2\Delta} - G_n \frac{\kappa}{r_n} F_n]$$

$$- F_n E F_n - G_n E G_n\}. \tag{24}$$

The potentials α/r_n, σr_n and κ/r_n are replaced by $\alpha/(n\Delta)$, $n\sigma\Delta$ and $\kappa/(n\Delta)$, respectively. Let us impose the normalization condition on F and G and replace the last terms $-F_nEF_n - G_nEG_n$ by $-E(F_nF_n - 1/N) - E(G_nG_n - 1/N)$, with N being the number of lattice (grid) points, and E is regarded as a Lagrange multiplier.

By variational calculus, Δ remains in all the terms including those with E, and all terms are divided by Δ. We then get the Dirac equation in the matrix form for the v-th eigenvector $\mathbf{x}^v$ and the associated eigenenergy E^v

$$H\mathbf{x}^v = E^v \mathbf{x}^v. \tag{25}$$

A detailed form of the equation is

$$\begin{bmatrix} H_A & H_B \\ H_C & H_D \end{bmatrix} \begin{bmatrix} \mathbf{F}^v \\ \mathbf{G}^v \end{bmatrix} = E^v \begin{bmatrix} \mathbf{F}^v \\ \mathbf{G}^v \end{bmatrix}, \tag{26}$$

where the components of the row vectors $\mathbf{x}^v$, $\mathbf{F}^v$ and $\mathbf{G}^v$ have the relations

$$x_i^v = F_i^v, \tag{27}$$

$$x_{N+i}^v = G_i^v, \tag{28}$$

for $1 \le i \le N$. It is to be noted that i is not the imaginary unit in complex numbers, but an integer. The matrix H has the following form

$$H = \begin{bmatrix} H_A & H_B \\ H_C & H_D \end{bmatrix}$$

$$= \begin{bmatrix} \times & 0 & 0 & 0 & \times & \times & 0 & 0 \\ 0 & \times & 0 & 0 & \times & \times & \times & 0 \\ 0 & 0 & \times & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & 0 & 0 & \times & \times \\ \times & \times & 0 & 0 & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & \times & 0 & 0 \\ 0 & \times & \times & \times & 0 & 0 & \times & 0 \\ 0 & 0 & \times & \times & 0 & 0 & 0 & \times \end{bmatrix}, \tag{29}$$

where for $1 \le i, j \le N$,

$$H_{Aij} = H_{ij}\delta_{ij} = (+m - \frac{\alpha}{i\Delta} + \sigma i\Delta)\delta_{ij}, \tag{30}$$

$$H_{Bij} = H_{i,N+j} = \begin{cases} \frac{-\kappa}{i\Delta} & \text{for } j = i \\ \frac{+1}{2\Delta} & \text{for } j = i-1 \\ \frac{-1}{2\Delta} & \text{for } j = i+1 \\ 0 & \text{for the others} \end{cases}, \tag{31}$$

$$H_{Cij} = H_{N+i,j} = \begin{cases} \frac{-\kappa}{i\Delta} & \text{for } j = i \\ \frac{-1}{2\Delta} & \text{for } j = i-1 \\ \frac{+1}{2\Delta} & \text{for } j = i+1 \\ 0 & \text{for the others} \end{cases}, \tag{32}$$

$$H_{Dij} = H_{N+i,N+j}\delta_{ij} = (-m - \frac{\alpha}{i\Delta} + \sigma i\Delta)\delta_{ij}. \tag{33}$$

3 Analytic eigenenergies of a fermion under a confining linear potential

3.1 Analytic eigenenergies of the Dirac equation with a linear potential

The quantum eigenenergies are now treated by considering the energies of a classical mechanical Hamiltonian for the principal properties of the Regge trajectory. We use the following matrix R in the same notation as the aforementioned matrix H

$$R = \begin{bmatrix} R_A & R_B \\ R_C & R_D \end{bmatrix}, \tag{34}$$

where

$$R_{p_r,q_r} = \frac{1}{C_{\mathrm{N}}}[\sin(\frac{2\pi p_r q_r}{2N+1})], \tag{35}$$

for $1 \le p_r, q_r \le 2N$, and C_{N} is a normalization constant and large for large N. We then have (emphasizing the matrix elements of R using parentheses [ ])

$$(HR)_{i,q_r} = \frac{1}{C_{\mathrm{N}}}\{$$

$$H_{ii}[\sin(\frac{2\pi(i)q_r}{2N+1})]$$

$$+ H_{i,N+i-1}[\sin(\frac{2\pi(N+i-1)q_r}{2N+1})]$$

$$+ H_{i,N+i}[\sin(\frac{2\pi(N+i)q_r}{2N+1})]$$

$$+ H_{i,N+i+1}[\sin(\frac{2\pi(N+i+1)q_r}{2N+1})]\}. \quad (36)$$

The second and last terms below "$1/C_{\rm N}\{$" in the above equation amount to an expression in term of the matrix component $[\sin(2\pi(N+i)q_r/(2N+1))]$ of R

$$H_{i,N+i-1}[\sin(\frac{2\pi(N+i-1)q_r}{2N+1})]$$

$$+ H_{i,N+i+1}[\sin(\frac{2\pi(N+i+1)q_r}{2N+1})]$$

$$= \frac{1}{2\Delta}[\sin(\frac{2\pi(N+i-1)q_r}{2N+1})]$$

$$+ \frac{-1}{2\Delta}[\sin(\frac{2\pi(N+i+1)q_r}{2N+1})]$$

$$= \frac{1}{\Delta}\sin(\frac{-2\pi q_r}{2N+1})\frac{\cos(\frac{2\pi(N+i)q_r}{2N+1})}{\sin(\frac{2\pi(N+i)q_r}{2N+1})}$$

$$\times [\sin(\frac{2\pi(N+i)q_r}{2N+1})]. \quad (37)$$

Thus, HR has been set to $H'R$ with

$$H'_{ii} = H_{ii}, \quad (38)$$

$$H'_{i-1,N+i} = 0, \quad (39)$$

$$H'_{i,N+i} = H_{i,N+i}$$

$$+ \frac{1}{\Delta}\sin(\frac{-2\pi q_r}{2N+1})\frac{\cos(\frac{2\pi(N+i)q_r}{2N+1})}{\sin(\frac{2\pi(N+i)q_r}{2N+1})}, \quad (40)$$

$$H'_{i+1,N+i} = 0, \quad (41)$$

where κ is contained in $H_{i,N+i}$.

In a similar way, we obtain

$$H'_{N+i,i-1} = 0, \quad (42)$$

$$H'_{N+i,i} = H_{N+i,i}$$

$$+ \frac{1}{\Delta}\sin(\frac{2\pi q_r}{2N+1})\frac{\cos(\frac{2\pi(i)q_r}{2N+1})}{\sin(\frac{2\pi(i)q_r}{2N+1})}, \quad (43)$$

$$H'_{N+i,i+1} = 0, \quad (44)$$

$$H'_{N+i,N+i} = H_{N+i,N+i}. \quad (45)$$

Therefore, the matrix H' has been reduced to

$$H' = \begin{bmatrix} \times & 0 & 0 & 0 & \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 & \times & 0 & 0 \\ 0 & 0 & \times & 0 & 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times & 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 & \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 & \times & 0 & 0 \\ 0 & 0 & \times & 0 & 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times & 0 & 0 & 0 & \times \end{bmatrix}. \quad (46)$$

Hamiltonian is Hermitian and $R^{\rm t} = R$, where $R^{\rm t}$ is the transpose of R. The result for the above matrix H' essentially implies that $HR = H'R = R^{\rm t}H^{\rm t} = R^{\rm t}H'^{\rm t}$. From $H' = R(R^{\rm t}H'^{\rm t})$ and $R^{\rm t}H'^{\rm t} = HR$, we get $H' = R(HR) = R^{\rm t}HR$, which implies that this process is a unitary transformation.

A set of four elements H'_{ii}, $H'_{i,N+1}$, $H'_{N+i,i}$ and $H'_{N+i,N+i}$ of the above matrix is independent of the other matrix elements. The matrix composed of these four matrix elements is diagonalized using a unitary matrix

$$U_{pq} = \begin{cases} u_{pq} & \text{for } p,q = i, N+i \\ \\ \delta_{pq} & \text{for the others} \end{cases}, \quad (47)$$

where u_{pq} are the four elements of the (two-dimensional) unitary matrix. The determinant to yield the eigenenergies is

$$(E - H'_{ii})(E - H'_{N+i,N+i}) - H'_{i,N+i}H'_{N+i,i} = 0. \quad (48)$$

We now consider physically meaningful phenomena, whose eigenenergies are those for the row with $q_r = 1$ (the lowest oscillation case) of the aforementioned matrix R, and drop the q_r-dependent term from the Hamiltonian owing to small $|\sin(\pm 2\pi q_r/(2N+1))|$. (The contribution of the string tension term $\sigma i\Delta$ to the energy Hamiltonian is large in the case of not small i within the strongly bound confinement regime, while $\kappa/(i\Delta)$ exhibits a singularity for low i.) Additionally, we neglect the masses of the composite fundamental fermions, for the cases in which the masses are small compared to the rotational energy. We then get the eigenenergies

$$(E + \frac{\alpha}{i\Delta} - \sigma i\Delta)^2 - (\frac{|\kappa|}{i\Delta})^2 = 0, \quad (49)$$

which states that

$$E = -\frac{\alpha}{i\Delta} + \sigma i\Delta + \frac{|\kappa|}{i\Delta}. \tag{50}$$

The above Coulomb term $-\alpha/(i\Delta)$ is disregarded for the extremely small Δ (in some sense beyond the regime of usual computer simulations with the larger Δ) due to the asymptotic freedom of the non-Abelian field, yielding

$$E = \sigma i\Delta + \frac{|\kappa|}{i\Delta}, \tag{51}$$

(i is not the complex number but an integer of the lattice index).

Here, we consider a function of the number x in the real continuum in the region $x > 0$, defined by

$$E_x = \frac{|k|}{x} + \sigma x = (\sigma x + \frac{|k|}{x}). \tag{52}$$

The function E_x above takes the minimum at $x_{\rm m} = (|\kappa|/\sigma)^{1/2}$ as

$$E_x^{\rm min} = \frac{|k|}{(|\kappa|/\sigma)^{1/2}} + \sigma(|\kappa|/\sigma)^{1/2}$$

$$= 2(|\kappa|\sigma)^{1/2}. \tag{53}$$

This $x_{\rm m}$ is (considering $N\Delta > x_{\rm m}$) measured with the lattice spacing Δ as

$$x_{\rm m} = i_{\rm m}\Delta + \epsilon_{\rm E}, \tag{54}$$

where $i_{\rm m}$ is an integer and $\epsilon_{\rm E}$ corresponds to a residual denoted as

$$-\frac{\Delta}{2} \leq \epsilon_{\rm E} < \frac{\Delta}{2}. \tag{55}$$

In the limit as $\Delta \to 0$, the residual $\epsilon_{\rm E}$ vanishes ($\epsilon_{\rm E} \to 0$), and the eigenenergy E in Eq. (51) at $i = i_m$ approaches the minimum value $E_{\rm min}$ equal to $E_x^{\rm min}$, giving

$$E_{\rm min}^2 = (E_x^{\rm min})^2 = 4|\kappa|\sigma, \tag{56}$$

(which is independent of Δ).

We note that $|\kappa|$ is the absolute value of the integer-type relativistic quantum number, which includes a sign and originates from the Dirac equation. It is noteworthy that only the secular equation gives rise to the absolute value.

We compare our equality Eq. (56) given above with that obtained from another theoretical method [24, 25] using the classical mechanical Hamiltonian in spherical coordinates, which is briefly summarized as follows. The relativistic classical mechanical Hamiltonian comprising the kinetic energy and a linear potential is denoted as

$$H^{\rm (cl)} = (P^2 + m^2)^{1/2} + \sigma r, \tag{57}$$

where P is the relativistic momentum of a fundamental particle. For small mass compared to the rotational energy, the classical mechanical Hamiltonian, which was described above, is reduced to

$$H^{\rm (cl)} = P + \sigma r. \tag{58}$$

Using the rotational quantity J, which roughly corresponds to $|\kappa|$ and which is written by

$$J = Pr, \tag{59}$$

the classical mechanical Hamiltonian $H^{\rm (cl)}$ amounts to

$$H^{\rm (cl)} = \frac{J}{r} + \sigma r. \tag{60}$$

The energy minimum $E_{\rm min}^{\rm (cl)}$ of $H^{\rm (cl)}$ also occurs at

$$r = (\frac{J}{\sigma})^{1/2}, \tag{61}$$

to give

$$(E_{\rm min}^{\rm (cl)})^2 = 4J\sigma. \tag{62}$$

The above relation essentially coincides with the aforementioned equality Eq. (56), which is consistent with the principal properties of the Regge trajectory [23].

From the experimentally observed slope

$$\frac{d|\kappa|}{d(E_{\rm min}^2)} = 0.93, \tag{63}$$

in natural units for the Regge trajectory [23, 28], the equality $|\kappa| = E_{\rm min}^2/(4\sigma)$ results in $1/(4\sigma) = 0.93$ GeV^{-2} to yield $\sqrt{\sigma} = 518.5$ MeV. If we use the relation $\sqrt{\sigma} = 2.255\Lambda_{\rm MOM}$, derived analytically in our previous paper [2] where $\Lambda_{\rm MOM}$ corresponds to the scale-invariant energy of quantum chromodynamics (QCD), we arrive at an $\Lambda_{\rm MOM}$ of 229.9 MeV. This is larger than the 186 MeV calculated for a smaller setting of $\sqrt{\sigma} = 420$ MeV in our previous paper [2]. These values are consistent with the observed QCD scale-invariant energy of around 213 MeV [6].

3.2 Discussions

Here, we add some discussions concerning the present approach and results obtained in the previous section and subsection. Unlike our approach, the classical mechanical Hamiltonian with a linear potential and repulsive rotational potential [24, 25], described by Eqs. (57)-(62) in this paper, has no basis in the Dirac/Yang-Mills equations. Furthermore, concerning the Regge trajectory expressed as a function of the rotational quantum number for eigenenergies (masses) of the pair of the constituent fundamental fermion and antifermion, the mechanism yielding large binding energies compared with the composite fermion masses has not yet been fully clarified by other theoretical/numerical approaches.

In contrast, we used the Dirac equation in spherical coordinates at the first quantization level, considering

the following points. First, because spherical coordinates differ from orthogonal coordinates, the eigenenergies are derived as a function of the relativistic quantum number, considering the expression of the Regge trajectory. Second, the procedure for deriving the solution of the Dirac equation shows that the mass (binding eigenenergy) of the pair of the fundamental fermion and antifermion has its origin in the linear force between the particles and the angular potential proportional to the relativistic angular quantum number. Therefore, the eigenequation in the form of determinant gives rise to the lowest energy as a function of the relativistic quantum number corresponding to the classical mechanical Hamiltonian energy for the principal properties of the Regge trajectory [24, 25], which is presented in the part containing Eqs. (57)-(62) in this article.

We note that our formalism for the Dirac equation in the present paper uses the linear potential with the attractive Coulomb potential, which was directly derived from the Yang-Mills equation using the path integral at the second quantized field-theoretic level. The potential used here was calculated non-perturbatively using the Wilson loop, which has all orders of boson contributions. The Wilson loop $W_{\rm Q}$ gave rise to (Coulomb potential + linear potential) in Eq. (7), as follows:

$$V_{\rm W}(r) = -\frac{\ln[W_{\rm Q}(r)]}{t_2 - t_1}, \tag{64}$$

where $t_2 - t_1$ is the time interval used in calculating the Wilson loop. If we use the Wilson loop derived in our previous paper [2], the linear force part, which was provided analytically, is expressed as

$$W_{\rm QL}(r) = \exp[-\sigma r(t_2 - t_1)], \tag{65}$$

where t_1 is a small quantity. Then, the detailed form of Eq. (50), when the rotational energy is large compared to the fermion mass, becomes

$$E = V_{\rm W}(i\Delta) + \frac{|\kappa|}{i\Delta}. \tag{66}$$

The eigenenergies of the Dirac equation obtained in this paper can be larger than the masses of the composite fermion and antifermion, when the masses are small compared to the rotational energy. The Polyakov line shows the deconfinement at high temperatures, and if we use the result of our previous paper [2], the Polyakov line P_τ which we analytically derived is expressed as

$$P_\tau = \cos\{\arccos[\exp(-\sigma r\tau)] - \arccos[\exp(-\sigma r\tau_\epsilon)]\}$$

$$\approx \cos\{\arccos[\exp(-\sigma r\tau)]\}, \tag{67}$$

with $\tau = 1/(k_B T)$ ($k_{\rm B}$ and T are the Boltzmann constant and temperature, respectively) and τ_ϵ being a small quantity. Then,

$$\epsilon_{\rm q} = -\ln(P_\tau) \approx -\ln[\exp(\frac{-E_{\rm B}}{k_{\rm B}T})] = \frac{E_{\rm B}}{k_{\rm B}T} = \frac{\sigma r}{k_{\rm B}T}, \tag{68}$$

where $\epsilon_{\rm q}$ is the binding energy of the pair of the fundamental fermion and antifermion, and $E_{\rm B} = \sigma r$. Equation (68) shows that $\epsilon_{\rm q}$ is small at high temperatures and the deconfinement of paired fermions occurs in some sense. The classical mechanical Hamiltonian describes the system at absolute zero (temperature) and did not treat this deconfinement at high temperatures.

Before the next discussions, we briefly refer to the quenched case, which takes into account the Okubo-Zweig-Iizuka (OZI) rule [29-31], implying that the further fermion-antifermion pair creation is suppressed. (The quenched case corresponds to the following approximation in the case of the path integral with respect to fermion Grassmann numbers $\bar{\Psi}$ and Ψ, including a matrix $M_{\rm f}$,

$$\int d\bar{\Psi} d\Psi \exp(-\bar{\Psi} M_{\rm f} \Psi) = \det(M_{\rm f}), \tag{69}$$

which is set to unity.)

In the operator formalism at Euclidian time t, the Green's (two-point correlation) function is given by

$$G_{t0}(t) = < 0|\hat{\mathcal{H}}_{\rm B}(t)\hat{\mathcal{H}}_{\rm B}^\dagger(0)|0 >, \tag{70}$$

where $< 0|$ and $|0 >$ is the ground state vacuum and $\hat{\mathcal{H}}_{\rm B}$ is the Heisenberg-type Hamiltonian operator of the bound state, which is expressed in terms of the energy operator $\hat{E}$ as

$$\hat{\mathcal{H}}_{\rm B}(t) = \exp(\hat{E}t)\hat{\mathcal{H}}_{\rm B}(0)\exp(-\hat{E}t). \tag{71}$$

Using Eqs. (20) and (21), the s-th solution of the Dirac equation $|s >$ with eigenenergy $E_{(s)}$ is denoted as

$$|s > \propto \begin{bmatrix} \frac{G_{(s)}}{r} \\ \\ \frac{F_{(s)}}{r} \end{bmatrix}. \tag{72}$$

(The above radial functions are multiplied by spin-angular components and a constant factor for the center of mass $\exp(iP_\mu X_\mu)$, with X_μ and P_μ being the position coordinates for the center of mass and its momentum, respectively.) For the quenched case, the Green's function given by Eq. (70) yields, using the above solution $|s >$,

$$G_{t0}(t) = \sum_s < 0|\hat{\mathcal{H}}_{\rm B}(t)|s >< s|\hat{\mathcal{H}}_{\rm B}^\dagger(0)|0 >$$

$$= \sum_s |< 0|\hat{\mathcal{H}}_{\rm B}(0)|s >|^2 \exp(-E_{(s)}t), \tag{73}$$

where the eigenenergy of the Dirac equation has appeared as a decay constant for the Euclidian time.

Thus, the solutions of the Dirac equation associated with the eigenenergies enter into the operator formalism with the help of the OZI rule (the suppression of the further fermion-pair creation). As previously mentioned,

the Dirac equation indicates that the lowest mass (expressed as an eigenenergy as a function of the relativistic quantum number when the rotational energy is larger than the constituent particle masses) of the pair of the fundamental fermion and antifermion originates in the confining linear potential and the angular potential proportional to the relativistic quantum number. The mass of the pair is allowed to be larger than the masses of the composite particles. Furthermore, the deconfinement feature at high temperatures in some sense is described by the Polyakov line according to the Yang-Mills theory, as presented in our previous paper [2].

4 Conclusions

We have presented a formalism for the Dirac field under a confining linear potential using basis functions localized in the spacetime continuum, which formulates fields of finite degrees of freedom. A given linear potential is that from the Wilson loop analysis for a non-Abelian Yang-Mills field. The Hamiltonian matrix has been analytically diagonalized with the use of two sequential unitary transformations, thus yielding the eigenenergies of the confined fundamental fermion-antifermion pair. The lowest eigenenergy (as a function of the relativistic quantum number for the large rotation energy compared to the composite particle masses) is proportional to the string tension and the Dirac's relativistic quantum number related to the total angular momentum, which is consistent with the expectation.

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